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equation (13) in Stangeby and Allen's (1970) paper. As before, we choose C to be a closed Mach line C_M (so that $q_n = -a$ and $\partial q_n/\partial s = 0$ on C_M) and integrate around C_M, yielding

$$\lim_{q_n \to -a} \oint_{C_M} \frac{(1-q_n^2/a^2)}{(1+q_s^2/a^2)} \frac{\partial q_n}{\partial n} \mathrm{d}s = 2\pi a + \lambda \oint_{C_M} \frac{\mathrm{d}s}{(1+q_s^2/a^2)} + (\nu+\lambda) \oint_{C_M} \mathrm{d}s. \qquad (6)$$

In general, the right-hand side of equation (6) is positive, so that

$$\lim_{q_n \to -a} \oint_{C_M} \frac{(1-q_n^2/a^2)}{(1+q_s^2/a^2)} \frac{\partial q_n}{\partial n} \mathrm{d}s > 0$$

and $\partial q_n/\partial n \to \infty$ on at least some portion of C_M. The analysis of Stangeby and Allen (1970) may now be applied to show that $\partial q_n/\partial n \to \infty$ everywhere on C_M.

Hence, the plasma–sheath boundary in a plasma with an anisotropic ion velocity distribution is a Mach line, i.e. the normal component of the ion velocity is equal to $\{k(T_e + T_i)/M\}^{1/2}$ irrespective of ionization or collision effects in the plasma.

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A possible mechanism for instability in a perpendicular collisionless shock wave

Abstract. Bernstein waves which propagate in a direction opposite to the current flow in a perpendicular collisionless shock can have negative energy. These negative-energy waves can give rise to instability either by coming into resonance with the ion acoustic wave or by dissipating their energy through ion Landau damping. In the second case the instability can take place for $T_i \gtrsim T_e$.

There has been much speculation about the nature of the instability which might occur within a collisionless shock wave propagating perpendicular to a strong magnetic field. Sagdeev (1966) has suggested an ion wave instability and this idea has recently been refined by Krall and Book (1969). Krall and Book considered waves propagating perpendicular to the magnetic field $\boldsymbol{B}_0$ and the shock front, which were driven unstable by the perpendicular drifts due to the gradients of magnetic field and density at the front. However, Gary and Sanderson (1970) have pointed out that there is a third source of drift, due to the voltage jump across the shock front. For low or moderate values of $\beta_e (\beta_e = n_0 \kappa T_e/(B_0^2/2\mu_0))$ the drift due to the voltage jump is the dominant one (Gary and Sanderson 1970).

The situation we wish to analyse is therefore the following. We assume the shock wave propagates in the x direction and take the z axis as the direction of the zero-order magnetic field. There is then a zero-order electric field in the x direction. The crucial assumption is that, since the time for the shock front to pass is much less than the ion Larmor period and much greater than the corresponding electron period, only the electrons undergo an $\boldsymbol{E}_0 \times \boldsymbol{B}_0$ drift. There is therefore a zero-order current flowing in the y direction.

Assuming the $\boldsymbol{E}_0 \times \boldsymbol{B}_0$ drift to be the dominant one we neglect the gradients in n_0 and $\boldsymbol{B}_0$ and consider electrostatic waves propagating at right angles to both $\boldsymbol{B}_0$ and the shock front, i.e. perturbations of the order of $\exp \mathrm{i}(ky-\omega t)$. We further assume that $\omega \gg \omega_{\mathrm{ci}}$ where ω_{ci} is the ion cyclotron frequency so that the ions are taken to be unmagnetized. With these assumptions one can easily obtain the dispersion relation:

$$1+\frac{\omega_{\mathrm{pi}}^2}{k^2 v_{T_\mathrm{i}}^2}\{1+\xi_\mathrm{i} Z(\xi_\mathrm{i})\}-\frac{\omega_{\mathrm{pe}}^2}{k^2 v_{T_\mathrm{e}}^2} \times\left\{-1+\mathrm{e}^{-\lambda}\mathrm{I}_0(\lambda)+2(\omega-kv_0)^2 \sum_{n=1}^{\infty} \frac{\mathrm{e}^{-\lambda}\mathrm{I}_n(\lambda)}{(\omega-kv_0)^2-n^2\omega_{\mathrm{ce}}^2}\right\}=0 \tag{1}$$

where ω_{pi} is the ion plasma frequency, v_{T_i} the ion thermal speed $(\kappa T_\mathrm{i}/m_\mathrm{i})^{1/2}$, $\xi_\mathrm{i} = \omega/\sqrt{2}\, kv_{T_\mathrm{i}}$ and Z the Fried–Conte (1961) function, and ω_{pe}, $\omega_{\mathrm{ce}}(=|e|B_0/m_\mathrm{e})$ and v_0 are respectively the electron plasma frequency, the electron cyclotron frequency and the electron drift velocity $-E_0/B_0$. The quantity λ is given by

$$\lambda=\frac{k^2\kappa T_\mathrm{e}}{m_\mathrm{e}\omega_{\mathrm{ce}}^2}=k^2\frac{v_{T_\mathrm{e}}^2}{\omega_{\mathrm{ce}}^2} \tag{2}$$

and I_n is the nth-order modified Bessel function of the first kind.

If $\omega \gg \omega_{\mathrm{pi}}$ the dispersion equation (1) becomes the equation for Bernstein (1958) waves in the presence of a Hall current. Krall and Book (1969) neglected these waves in their analysis but Gary and Sanderson (1970) have shown that they can have a profound effect on the stability of the system. The reason the Bernstein waves (at harmonics of the electron cyclotron frequency) are important, and this was not pointed out by Gary and Sanderson, is that in the presence of a Hall current they can have negative energy. We can see this quite simply from their dispersion relation. The small signal energy density of an electrostatic wave is given by

$$\mathscr{E}=\tfrac{1}{4}\epsilon_0|E|^2\frac{\partial}{\partial\omega}\{\omega\epsilon_1(\omega,k)\}_{\epsilon_1=0} \tag{3}$$

where $\epsilon_1(\omega,k)$ is the longitudinal dielectric constant and is given by

$$\epsilon_1(\omega,k)=1-\frac{\omega_{\mathrm{pe}}^2}{k^2 v_{T_\mathrm{e}}^2}\left\{-1+\mathrm{e}^{-\lambda}\mathrm{I}_0(\lambda)+2(\omega-kv_0)^2\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-\lambda}\mathrm{I}_n(\lambda)}{(\omega-kv_0)^2-n^2\omega_{\mathrm{ce}}^2}\right\}. \tag{4}$$

From equations (3) and (4) it follows that the condition for $\mathscr{E} < 0$ is

$$0<\omega<kv_0. \tag{5}$$

We now return to the general dispersion equation given by equation (1). Owing to the negative energy property of the Bernstein waves there will be an instability analogous to the two-stream instability when an ion wave becomes degenerate with a

Bernstein harmonic (i.e. when two branches merge at some critical point ω_0, k_0 in the dispersion diagram). We can demonstrate this instability with the aid of a method much used in the theory of travelling wave tubes (Pierce 1950). For $T_i \ll T_e$, equation (3) becomes

$$1-\frac{\omega_{pi}^2}{\omega^2}-\frac{\omega_{pe}^2}{k^2 v_{T_e}^2}\left\{-1+e^{-\lambda}I_0(\lambda)+(\omega-kv_0)^2\sum_{n=1}^{\infty}\frac{2e^{-\lambda}I_n(\lambda)}{(\omega-kv_0)^2-n^2\omega_{ce}^2}\right\}=0. \tag{6}$$

The instability will only occur when one of the Bernstein harmonics 'intersects' the ion wave. This will be for large values of k where we can approximate the infinite sum by one harmonic provided

$$\omega-kv_0 \simeq \pm n\omega_{ce} \tag{7}$$

and we can therefore write equation (6) in the form

$$(\omega^2-k^2c_s^2)\{(\omega-kv_0)^2-n^2\omega_{ce}^2\} = 2\omega^2(\omega-kv_0)^2 e^{-\lambda}I_n(\lambda) \tag{8}$$

where we have used the fact that $\omega_{ce}/kv_{T_e} \ll 1$. Under these conditions the right-hand side of equation (8) is much less than unity and we can use a perturbation technique to obtain a solution. Instability results from a resonance between two branches of the dispersion relation. Since it is the slow Bernstein mode which carries negative energy the resonance condition is the following:

$$kc_s \simeq kv_0-n\omega_{ce} \tag{9}$$

where we take $v_0 > 0$. We now look for solutions

$$\omega = kv_0-n\omega_{ce}+\delta\omega \tag{10}$$

and obtain

$$(\delta\omega)^2 = -\frac{1}{2(2\pi)^{1/2}}n\omega_{ce}^2\left(\frac{m_e}{m_i}\right)^{1/2} \tag{11}$$

where we have used the result

$$I_n(\lambda) \simeq \frac{e^{\lambda}}{(2\pi\lambda)^{1/2}} \qquad \text{for} \qquad \lambda \gg 1.$$

The right-hand side of equation (11) is negative definite and so we have instability. The growth rate is

$$\frac{\gamma}{\omega_{ce}} = \frac{n^{1/2}}{(8\pi)^{1/4}}\left(\frac{m_e}{m_i}\right)^{1/4} \tag{12}$$

and the values of k at which there is instability are given from equation (9). They are

$$k\rho_e = \frac{nv_{Te}}{(v_0-c_s)} \tag{13}$$

where ρ_e is the electron Larmor radius.

The condition for the validity of the solution is that

$$k\rho_e \gg \frac{n^{1/2}}{(8\pi)^{1/4}}\left(\frac{m_i}{m_e}\right)^{1/4}. \tag{14}$$

Gary and Sanderson (1970) obtained a numerical solution for this case. However, there is second instability mechanism which was not considered by Gary and Sanderson (1970). This can only be understood from the point of view of a negative-energy wave.

Whenever a system propagates a negative-energy wave instability results when there is a sink to which the wave can give up its energy. In the above example the sink was provided by a positive-energy ion wave. However, for a plasma with hot ions there is an alternative sink. Those ions whose thermal velocities are of the order of the phase velocity of the negative energy wave can absorb its energy and enable it to grow. Thus, for a negative-energy wave, 'Landau damping' due to another species produces instability! To demonstrate this effect we again return to equation (11). First, we would expect the most marked effect for frequencies such that

$$\omega \sim kv_{T_\mathrm{i}}. \tag{15}$$

For $T_\mathrm{e} > T_\mathrm{i}$ such frequencies are well separated from the ion–acoustic frequencies and so there would be no instability of the type considered above. We again assume that a particular harmonic in the infinite summation is dominant. The dispersion relation can now be written:

$$\left[1+\frac{\omega_{\mathrm{pi}}^2}{k^2v_{T_\mathrm{i}}^2}\{1+\xi_\mathrm{i}Z(\xi_\mathrm{i})\}+\frac{\omega_{\mathrm{pe}}^2}{k^2v_{T_\mathrm{e}}^2}\right]\{(\omega-kv_0)^2-n^2\omega_{\mathrm{ce}}^2\}$$
$$=2(\omega-kv_0)^2\frac{\omega_{\mathrm{pe}}^2}{k^2v_{T_\mathrm{e}}^2}\mathrm{e}^{-\lambda}\mathrm{I}_n(\lambda). \tag{16}$$

Looking for solutions which satisfy

$$\omega = \sqrt{2}kv_{T_\mathrm{i}}$$

we can substitute this into the first bracket, on the left-hand side of equation (16), to obtain

$$\left\{1-\frac{\omega_{\mathrm{pi}}^2}{k^2v_{T_\mathrm{i}}^2}\mathrm{e}^{-1}(1-\mathrm{i}\pi^{1/2})\frac{\omega_{\mathrm{pe}}^2}{k^2v_{T_\mathrm{e}}^2}\right\}\{(\omega-kv_0)^2-n^2\omega_{\mathrm{ce}}^2\}$$
$$=2(\omega-kv_0)^2\frac{\omega_{\mathrm{pe}}^2}{k^2v_{T_\mathrm{e}}^2}\mathrm{e}^{-\lambda}\mathrm{I}_n(\lambda). \tag{17}$$

We can now see that the ions are behaving as a medium of complex dielectric constant (Birdsall *et al.* 1953):

$$1-\frac{\omega_{\mathrm{pi}}^2}{k^2v_{T_\mathrm{i}}^2}\mathrm{e}^{-1}(1-\mathrm{i}\pi^{1/2}).$$

For $k^2v_{T_\mathrm{i}}^2/\omega_{\mathrm{pi}}^2 \ll 1$, approximating the right-hand side as before and taking $T_\mathrm{i} \ll T_\mathrm{e}$, we can simplify equation (17) even more:

$$\left(\frac{\omega-kv_0}{n\omega_{\mathrm{ce}}}\right)^2-1=\frac{2}{(2\pi)^{1/2}}\frac{T_\mathrm{i}}{T_\mathrm{e}}\frac{1}{k\rho_\mathrm{e}}\frac{\mathrm{e}^{-1}}{(-1+\mathrm{i}\pi^{1/2})}. \tag{18}$$

The solution of equation (18) is

$$\omega = kv_0 \pm n\omega_{\mathrm{ce}}\left\{1-\frac{\mathrm{e}^{-1}}{(2\pi)^{1/2}}\frac{(1+\mathrm{i}\pi^{1/2})}{(1+\pi)}\frac{T_\mathrm{i}}{T_\mathrm{e}}\frac{1}{k\rho_\mathrm{e}}\right\}. \tag{19}$$

We see that the slow Bernstein wave (i.e. the negative-energy wave) is growing in time. For the validity of the solution (19) we require

$$kv_0-n\omega_{\mathrm{ce}} = \sqrt{2}kv_{T_\mathrm{i}} \tag{20}$$

and

$$k^2\rho_\mathrm{e}^2 \gg \frac{3}{8}\left(\frac{T_\mathrm{i}}{T_\mathrm{e}}\right)^{1/2}\left(\frac{m_\mathrm{i}}{m_\mathrm{e}}\right)^{1/2}. \tag{21}$$

We have considered this instability when its growth rate is a maximum (i.e. when the phase velocity of the slow Bernstein wave is of the order of the ion thermal velocity). For $T_i \ll T_e$ the range of k values for which a given harmonic becomes unstable is small. For a typical value $v_0/v_{T_e} = 0{\cdot}1$ (Paul *et al.* 1965) these k values can be obtained from equation (20):

$$k\rho_e \simeq 12n. \tag{22}$$

For $T_e/T_i = 5$ the growth rates of each harmonic can be obtained from equation (19) giving

$$\frac{\gamma}{\omega_{ce}} = 0{\cdot}008 \tag{23}$$

independent of n.

This instability does not require T_i to be small compared with T_e since it depends only on resonant ions. As T_i increases the range of unstable k values increases (for each harmonic). Returning to equation (17) we can obtain the growth rate due to this instability for arbitrary values of T_i:

$$\frac{\gamma}{\omega_{ce}} = \frac{e^1 nT_i/\surd 2T_e k\rho_e}{\pi + \{(T_i/T_e)e^1 - 1\}^2}.$$

For $T_i = T_e$, $k\rho_e = 15$ and $v_0/v_{T_e} = 0{\cdot}1$ we have

$$\frac{\gamma}{\omega_{ce}} = 0{\cdot}02.$$

The two significant points about this instability are first that it results in energy being absorbed by the ions (i.e. the power is positive), and secondly that it can take place for $T_i \gtrsim T_e$. For large values of T_e/T_i it would probably be swamped by the faster growing instability described in the first part of this paper.

In the above we have considered two instability mechanisms which may occur within a perpendicular collisionless shock. Both mechanisms are due to the negative energy character of the Bernstein modes in the presence of an $\boldsymbol{E}_0 \times \boldsymbol{B}_0$ drift on the electrons. The first results from a resonance between the slow Bernstein wave and the ion acoustic wave and the second is due to the Landau damping of the ions. The resonant ions absorb energy from the slow Bernstein wave thus causing it to grow, i.e. the energy from the instability goes directly to the ions. This second instability can occur for arbitrary values of T_i/T_e and in particular for $T_i \gtrsim T_e$. A fuller account of this work in which the gradients in density and magnetic field are included will be published later. In fact, similar results are obtained but the density gradient results in there being a maximum value of k above which there is no instability.

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The effects of spatial coherence on intensity fluctuation distributions of Gaussian light

Abstract. The effects of finite sizes of source and detector on observed intensity fluctuations of Gaussian light are investigated both theoretically and experimentally. The theory allows a measurement of source size to be made using a single receiving aperture, rather than two as used in the Hanbury–Brown Twiss intensity interferometer.

In much recent work in statistical optics, temporal coherence properties have been investigated. Theoretical and experimental results have been obtained in which the effects of spatial coherence have been considered to be negligibly small due to the use of essentially point sources and detectors. A recent bibliography is given by Jakeman and Pike (1969). In real experiments, of course, the order of size of a 'point' source or detector before spatial coherence effects become measurable is of interest. Conversely, effects due to loss of spatial correlation can be used, as in the Michelson stellar interferometer, or the Hanbury–Brown Twiss intensity interferometer, to provide information about the source.

In recently reported quantitative work from this laboratory using intensity fluctuation spectroscopy of laser scattering to determine diffusion constants of protein molecules (Foord *et al.* 1970) it was necessary to calculate the effects of finite aperture sizes on the measurements in order to interpret fully the results obtained. This calculation together with supporting experimental data are presented in this paper. The results are applicable to the general problem of spatial integration over a detector surface of light from a quasimonochromatic Gaussian source. We prove this first below by showing that the mutual coherence of the scattered laser field between two points on the detector surface is identical to that arising from such a source. We then calculate the second moment of the intensity fluctuation distribution measured by a single detector, whose area is not small compared with a 'coherence area'. The result, which involves simply a double integral of the square of the Van Cittert–Zernicke mutual coherence function over the detector surface, can be thought of as providing the theory of a single-aperture intensity interferometer.

Consider the electric field at the point $\boldsymbol{r}$ due to the scattering of laser light of wave vector $\boldsymbol{k}_0$ from particles situated at the point $\boldsymbol{r}_j$ and moving with velocity $\boldsymbol{v}_j$. Let a fraction $n_\omega \delta\omega$ of the particles give rise to frequency shifts at $\boldsymbol{r}$ between ω and $\omega + \delta\omega$. The formula

$$\omega = \omega_0 - \boldsymbol{K}_j(\omega) \,.\, \boldsymbol{v}_j \tag{1}$$